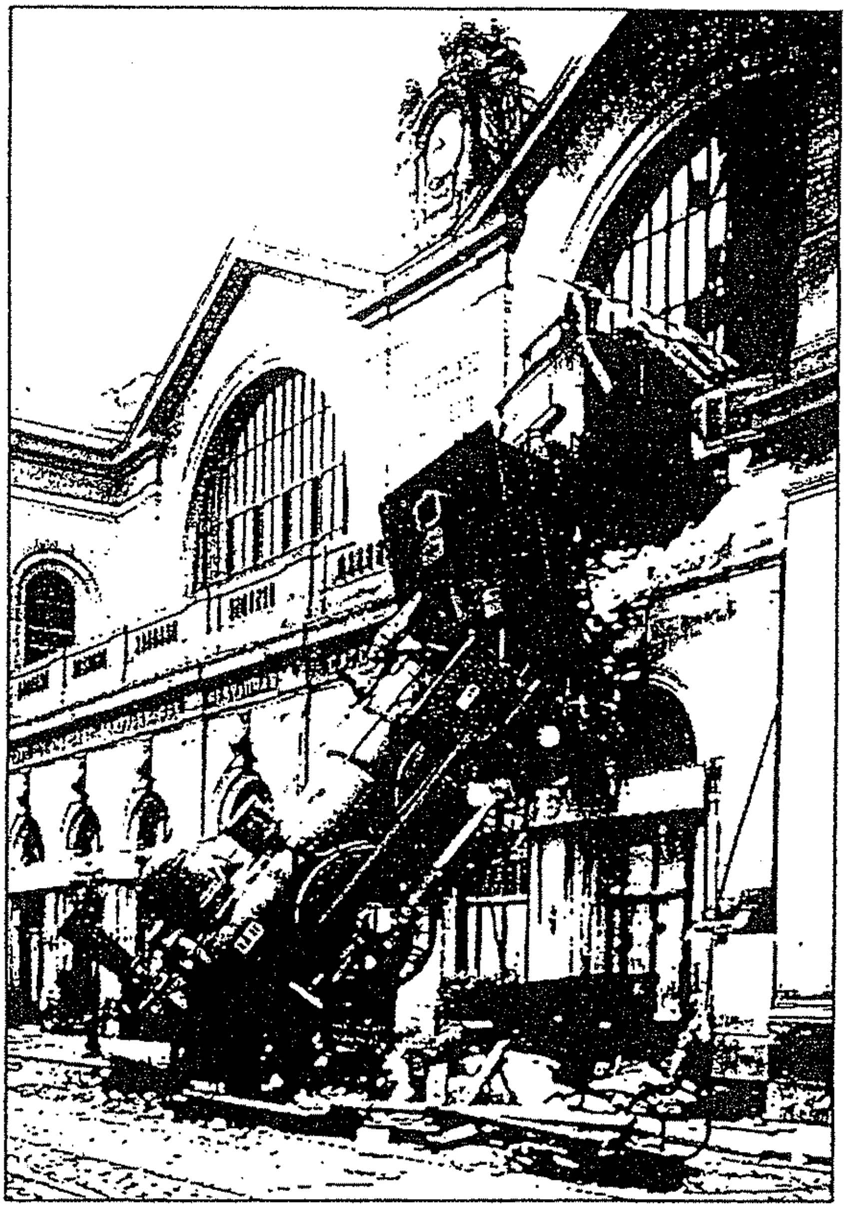
I.B. PHYSICS STUDENT BOOKLET

DEALING WITH UNCERTAINTIES



*Name & Class Code*____

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Unless you can measure what you are speaking about and express it in numbers, you have scarcely advanced to the stage of science.

—Lord Kelvin

PHYSICS is a quantitative science—it deals with numbers and measurements. If we are to understand the knowledge of physics we must be aware of the *nature* of measurement. When the purpose of an experiment is to test a theory, we ask how accurate and how precise the predictions need to be. If the theory predicts a deflection of light by 12° but experiment reveals an angle of only 10°, is the theory then confirmed or rejected? The importance of errors and uncertainties in physics cannot be overstated.

The purpose of *Dealing With Uncertainties* is to help you appreciate the quality of your experimental work. This booklet explains the basic treatment of uncertainties as it can be practiced in the IB Physics curriculum. A more mathematical approach is possible, including statistical analysis, but for a high school survey course the treatment suggested here is more than adequate.

INTRODUCTION. There is a basic difference between counting and measuring. My class has exactly 16 students in it, not 15.5 or 15.4. That's counting. But a given student is never exactly 6 feet tall, nor is she 6.000 feet tall. There is always some limit to the accuracy and precision in our knowledge of any measured property—the student's height, the time of an event, the mass of a body. **Measurements always contain a degree of uncertainty.** Appreciating the uncertainty in laboratory work will help demonstrate the *reliability* and the *reproducibility* of the investigation, and these qualities are hallmarks of any good science.

Why do measurements always contain uncertainties? Physical quantities are never perfectly defined and so no measurement can be expressed with an infinite number of significant figures; the so called 'true' value is never reached. There are also hidden uncertainties which are part of the measurement technique itself, such as systematic or random variations. The resolution of an instrument is never infinitely fine, analogue scales need to be interpreted, and instruments themselves need calibration. All this adds to the uncertainty of measurement. We can reduce uncertainty but we cannot escape it. In order to do good science, we need to acknowledge these limits.

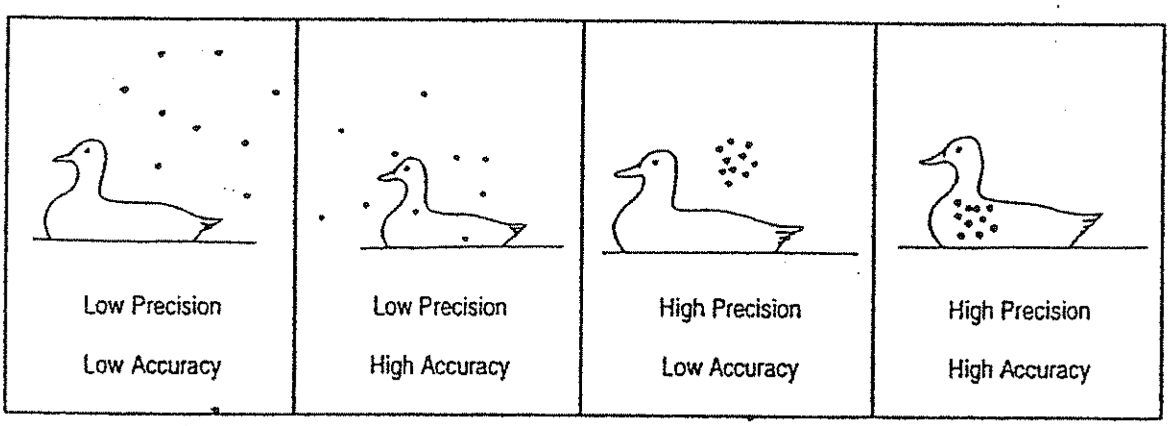
PRECISION AND ACCURACY. To obtain a more reliable result, a physical quantity is often measured a number of times. For instance, the period of a pendulum may be measured (in seconds) three times to be 3.3, 3.2, 3.3. There is a relatively small variation or range of measurements here, and so we say the precision is high. But if the results have a relatively large range, say 3.3, 2,8, 3.5, then the we say precision is low. The **precision** of a series of measurements is an indication of the agreement among repetitive measurements. Precision can be quantified as the standard deviation of the measured values, but

this is not necessary in IB Physics work. An example of an uncertainty due to the *lack of precision* would be the measurement of the thickness of a thin wire using a meter stick. The wire is only a fraction of a millimeter thick while the meter stick is calibrated in millimeters, and so the precision of the measurement will be low.

The number of justifiable significant figures also expresses a notion of **precision**. Two students may have calculated the free-fall acceleration due to gravity as $9.625 m s^{-2}$ and $9.8 m s^{-2}$ respectively. The former is more precise—there are more significant figures—but the latter value is more accurate; it is closer to the correct answer. Precision, then, can mean the resolution of measurements, be it meters or millimeters. An amazing example of precision is the measured charge of an electron, $q_e = (1.6021773 \pm 0.0000005) \times 10^{-19} C$, which represents an uncertainty of only $3 \times 10^{-5}\%$. Another example of high precision is the measured equivalence of inertial and gravitational mass, where $m_i \doteq m_g$ is good to 1 part in 10^{12}, an uncertainty of only $10^{-10}\%$.

The **accuracy** of a measurement is its relation to the *true*, *nominal*, or *accepted* value. This can be expressed as a percentage deviation from the known value. In most cases, the true value is an experimental value, such as the charge of an electron, and acceptance is based upon reproducible measurements. An example of an uncertainty due to the *lack of accuracy* would be the measured length of a football field using only a single-meter stick, laying it end on end. Although the meter stick has a precision of one millimeter, your overall measurement would be far from accurate. In physics, we seek both precision and accuracy; it is of little value to have a digital stopwatch displaying time in thousands of a second if the electrical circuitry is running slow.

Consider a hunter shooting ducks. Don't worry, the ducks are plastic. The four figures sketched below represent combinations of precision and accuracy.



We can conclude that an *accurate* shot means we are close to (and hit) the target but the uncertainty could be of any magnitude, large or small. To be *precise*, however, means there is a small uncertainty,

but this does not mean that we hit the target. To be both *accurate and precise* means we hit the target often and have only a small uncertainty.

UNCERTAINTIES AND/OR ERRORS. Some textbooks treat *errors* as equivalent to *uncertainties*. Errors, of course, are mistakes, and you can often find your mistake and correct it. Human errors arise from carelessness. Perhaps you misread a meter, writing down 1.50 instead of 1.05. Repeated readings can reveal this. An error can also be the difference between an accepted value and an experimentally measured value. Perhaps you calculated gravity as $(9.6 \pm 0.1) ms^{-2}$ whereas the accepted value is $9.81 ms^{-2}$. Your estimated uncertainty is $\pm 0.1 ms^{-2}$ and gives you an probable range from 9.5 to 9.7 ms^{-2}. But the *error* here is the difference between your calculation and the accepted value, an error of $0.2 ms^{-2}$. Clearly, the error was not an explicit part of your uncertainty, and your result is off due to errors and uncertainties not yet accounted for. In contrast to errors, an **uncertainty** is a limit to the precision of a measurement or calculation. In this booklet I will keep the terms 'error' and 'uncertainty' distinct.

SIGNIFICANT FIGURES. You first become aware of uncertainties when you deal with significant figures. There may be no mention of errors or uncertainties in a given calculation, but you must still decide on the number of significant figures you quote in your final answer. Consider the calculation of the circumference of a circle, where $c = 2\pi r$ and the radius $r = 4.1 cm$. Enter these quantities into your calculator and press the equals key, and the solution is given as $c = 25.76106 cm$. What does the 0.00006 in the calculation really mean? Do we know the circumference to 6 one-hundred thousandths of a centimeter? The 2 in $2\pi r$ is an integer and we assume it has infinite accuracy and zero uncertainty, and π can be quoted to any degree of precision and we also assume it is known accurately. But the radius is given to only two significant figures, and this represents a limit on the precision of any calculation using this value. The circumference is known, therefore, to two significant figures, and we can only say with confidence that $c = 26 cm$.

There are some general rules for determining significant figures. (1) The leftmost non-zero digit is the most significant figure. (2) If there is no decimal point, the rightmost non-zero digit is the least significant figure. (3) If there is a decimal point, the rightmost digit is the least significant digit, even if it is a zero. (4) All digits between the most significant digit and the least significant digit are significant figures. For instance, the number "12.345" has five significant figures, and "0.00321" has three significant figures. The number "100" has only one significant figure, whereas "100." has three. Scientific notation helps clarify significant figures, so that 1.00×10^3 has three significant figures as does 1.00×10^{-3}, but 0.001 or 1×10^{-3} have only one significant figure.

Significant figures reflect precision. The result of any calculation cannot improve upon the number of significant figures. A rule can be stated here. **The quantity with the least number of significant figures tells you the number of significant figures you may give in your final answer.** The product of $22 \times 0.10145 = 2.2319$ must be rounded off to two significant figures, so we can only say that $22 \times 0.10145 \approx 2.2$.

DEALING WITH UNCERTAINTIES. To help understand the technical terms used in our treatment of uncertainties, consider an example where a length of string is measured to be 24.5 cm long, or $\ell = 24.5 cm$. This *best measurement* is called the **absolute value** of the measured quantity. It is 'absolute' not because it is forever fixed but because it is the raw measured value without any appreciation of uncertainty. Next, we estimate the **absolute uncertainty** in the measurement, appreciating that the string is not perfectly straight, and that at both the zero and measured end of the ruler there is some interpretation of the scale. This is 'absolute' because it is a raw value, with units the same as the measured or calculated quantity. Perhaps we estimate the uncertainty to be 0.2 cm; we say that the absolute uncertainty here is $\Delta \ell = 0.2 cm$ (where Δ is pronounced 'delta'). A repeated measurement or a more precise measurement of the string might reveal it to be slightly longer or slightly shorter than the initial absolute value, and so we express the uncertainty as "plus or minus the absolute uncertainty." The length and its uncertainty is $\ell \pm \Delta \ell = 24.5 cm \pm 0.2 cm = (24.5 \pm 0.2) cm$.

We now understand the string's length measurement by saying that there is a range of **probable values.** The minimum probable value is $\ell_{min} = (24.5 - 0.2) cm = 24.3 cm$ and the maximum probable value is $\ell_{max} = (24.5 + 0.2) cm = 24.5 cm$. By using the parentheses in (24.5 ± 0.2) we can easily match the uncertainty value with the least significant figure in the absolute value. Uncertainty is rarely needed to more than one significant figure. Therefore, we can state a guideline here. **When adding experimental uncertainty to measured or calculated values, uncertainties should be rounded to one significant figure.** We might say ± 6 or ± 0.02 but we should not say ± 63.5 or ± 0.015. Also, we cannot expect our uncertainty to be more precise than the quantity itself because then our claim of uncertainty would be insignificant. Therefore there is another guideline. **The last significant figure in any stated answer should be of the same order of magnitude (in the same decimal position) as the uncertainty.** We might say 432 ± 3 or 3.06 ± 0.01 but not 432 ± 0.5 or 0.6 ± 0.02.

REPEATED MEASUREMENTS. A statistical approach to a number of measurements of the same quantity will reveal a non-uniform scatter of values about an average value. The further away from this average, the less likely you are to find another value. The distribution is something like a bell-curve. It is thus unlikely that the range from the minimum to the maximum individual values represents the

reasonable range of uncertainty. Repeated measurements usually do not increase the range but do help reinforce a mean value. Therefore, we want to reduce the probable range of uncertainty and give more credit for the values near a mean value. Consider the average or *mean* ℓ_{mean} of four independent measurements of a certain length ℓ, where $\ell_1 = 140\,cm \pm 2\,cm$, $\ell_2 = 136\,cm \pm 2\,cm$, $\ell_3 = 142\,cm \pm 1\,cm$, and $\ell_4 = 144\,cm \pm 5\,cm$.

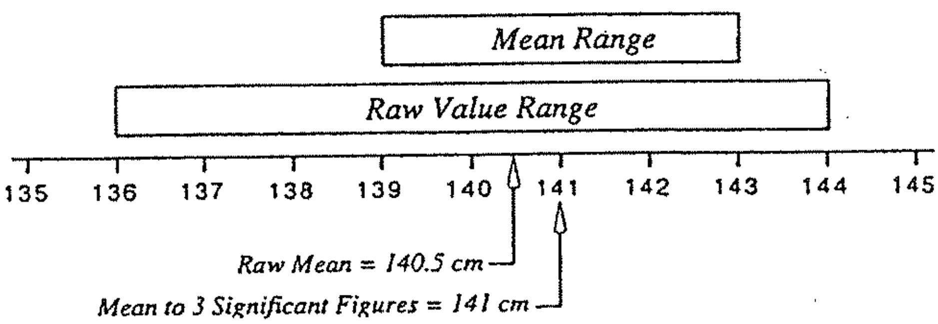
$$\ell_{mean} = \frac{\ell_1 + \ell_2 + \ldots + \ell_n}{n} = \frac{140\,cm + 136\,cm + 142\,cm + 144\,cm}{4} = 140.5\,cm$$

A reasonable expression of the uncertainty of the mean may be found by dividing the range (the difference between the largest and the smallest values in the data) by the number of measurements.

$$Uncertainty\ in\ Mean = \frac{\ell_{max} - \ell_{min}}{n} = \frac{144\,cm - 136\,cm}{4} = 2.00\,cm$$

The mean and its uncertainty can now be expressed as $\ell \pm \Delta\ell = 140.5\,cm \pm 2.00\,cm$ or, in correct form, as $\ell \pm \Delta\ell = (141 \pm 2)\,cm$. Notice how the range of absolute values goes beyond the uncertainty of the mean. We can improve upon the uncertainty range of an average value but there is no guarantee of improving accuracy with repeated measurements. Moreover, we can not improve on the precision of any measurements. A series of measurements with 3 significant figures cannot yield a mean value with 4 or 5 significant figures. This simplified method is not the same as the statistical calculation of mean deviation (where you find the average of the differences between each value and the mean).

Number Line Showing Uncertainty Range for Mean and Raw Values



REJECTING DATA. Sometimes one measurement in a series of repeated measurements of the same quantity appears to disagree with the pattern of the others. When this happens, you must decide whether the suspect measurement resulted from some mistake and should be rejected, or was a bona fide measurement and should be used with all the other data. For example, imagine you make measurements of the

period of a pendulum and get the results (in seconds) of 3.8, 3.5, 3.9, 3.9, 3.4, 1.8. The value 1.8 is noticeably different from the others, and we must decide what to do with it.

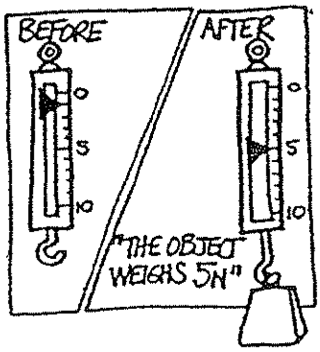
If you cannot find the cause for the suspect result you must then decide whether or not to reject it by examining the other measurements. In our example, the best estimate of the period of the pendulum is significantly affected if we reject the suspect 1.8 seconds. The average of all six measurements is about 3.4 seconds, whereas that of the first five is 3.7 seconds. The decision to reject data is ultimately a subjective one, but one which requires careful judgment. Sometimes drawing a graph helps you appreciate the degree of how inconsistent a given measurement is. The best straight line graph is drawn using only the consistent data points. You should include a suspect data point but circle it and indicate that you ignored it when drawing the best straight line.

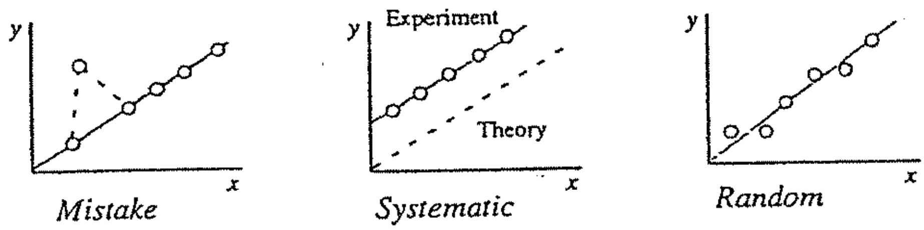
COUNTING MEASUREMENTS. Some experiments involve counting events that occur at random but with a definite average rate. For example, in a sample of radioactive material, each individual nucleus decays at a random time, but there is a definite average rate at which we would expect to see decays in the whole sample. We can try to measure this average rate by observing how many decays occur within some finite interval, such as one minute. Suppose we find that N decays have occurred after one minute. Because the decays occur at random, we can ask just how reliable N is as a measure of the expected average number of events. The answer is that the uncertainty in N is $\pm\sqrt{N}$, and so the average number of events is $N \pm \sqrt{N}$. If we count 15 decays from a radioactive sample over a time of one minute, we would conclude that, on average, our sample undergoes $15 \pm \sqrt{15} = 15 \pm 3.8972 = 15 \pm 4$ decays per minute. The probable range for any given minute would be from 11 to 19 decays per minute.

RANDOM AND SYSTEMATIC UNCERTAINTIES. A series of measurements may be recorded with the greatest possible precision but the numbers will often differ. For example, in measuring the thickness of a piece of wire with a micrometer screw gauge, the wire may not be of uniform thickness, the jaws of the gauge may be closed differently each time, or there may be temperature variations between measurements. There are a variety of reasons for imperfect measurements.

The most common are **random uncertainties**; these occur as deviations from a normalized value. Repeated measurements of the same quantity are just as likely to be either too small or too large. Random variations could be caused by such things as slight changes in atmospheric pressure, room temperature, supply voltage, or by changes of friction or pulling force when a trolley is pulled down different parts of a runway. Uncertainties in instrument readings can also be random; you may misread the scale, or read from the wrong end of a meter stick. The size of random uncertainties depends in part on the skill of the experimenter. Two different students doing the same laboratory experiment often get different results. As the skill of the experimenter increases, so the random errors and uncertainties should reduce.

When a series of measurements is consistently shifted in one direction we say there is a **systematic uncertainty** in these results. An experiment may contain a systematic fault when, for instance, an ammeter is not correctly zeroed and so gives consistently low readings. A stray magnetic field may offset all measurements, or the background count in a radioactivity measurement needs to be accounted for. A spring balance [see the sketch] may not have been correctly zeroed and so resulting measurements are incorrect. Repeated measurements will have no effect on systematic errors but graphing your data can reveal the consistent shift in the data. To eliminate systematic uncertainties introduced by a measuring instrument, we must calibrate it using standards that are known to be of high precision and accuracy.





Mistakes or personal errors are clearly inconsistent with the rest of the data. Systematic errors are consistent but shifted away from the theoretical line (in this case, the theory is $y = mx + c$ where $c = 0$). Random uncertainties, the most common and inevitable limit to our data, gives a statistical scatter.

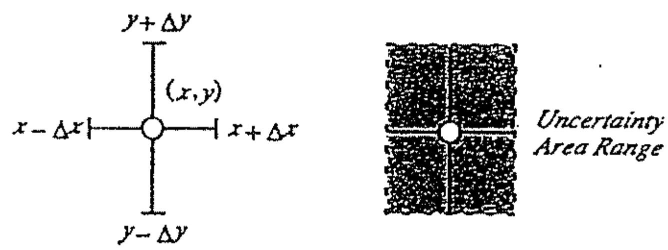
There is the possibility of an error inherent in some digital equipment. Occasionally digital measuring devices will introduce an uncertainty called **quantisation error**. This is caused by the conversion of measurements from analogue (or a continuous signal) to digital representations of that signal. A number of measurements are made over a time interval. The signal source may vary during the sampling time. As a result, the true signal and the measured signal may differ. This is especially true if the source signal varies at a frequency higher than the sampling frequency. In odd cases, a beat frequency might occur and totally wrong result can obtained. In the vast majority cases, however, this does not happen, and the sampling is representative of the source itself.

ANALOGUE AND DIGITAL MEASUREMENTS. Reading analogue scales requires interpretation. Measuring the length of a pencil against a ruler with millimeters divisions requires judgments about the nearest millimeter or fraction of a millimeter. For an analogue scale, we can usually detect with confidence one-half the smallest division at both the measured end and the zeroed end. We can say that our measurement has an uncertainty of plus or minus the smallest division. If the scale divisions are large, we can often interpolate even smaller divisions.

Digital readouts are not scales but are displays of integers, such as 1234 or 0.0021. Here, no interpretation or judgment is required, but we should not assume there is no uncertainty. There is a difference between 123 and 123.4, and so digital readouts are limited in their precision by the number of digits they display. A voltage display of 123 *V* could be the response to a potential difference of 122.9 *V* or 123.9 *V*, or any voltage within a range of about one volt. Although there is no interpolation with a digital readout there is still an uncertainty. The displayed value is uncertain to at least plus or minus one digit of the last significant figure (the smallest unit of measure). There may be, of course, other reasons affecting the uncertainty that would increase this value, such as the calibration, the sampling method, a systematic error or a linearity problem.

UNCERTAINTIES IN GRAPHS. Too often students will draw a graph by connecting the dots. Not only does this look bad, it keeps us from seeing the desired relationship of the graphed physical quantities. Connecting data-point to data-point is wrong. Even when there is no measurable uncertainty, I have students mark data points with small circles or crosses. When a graph includes uncertainty limits, or **uncertainty bars**, the best straight line more easily connects to the data regions. With an uncertainty bar, the data 'point' becomes a data 'area.'

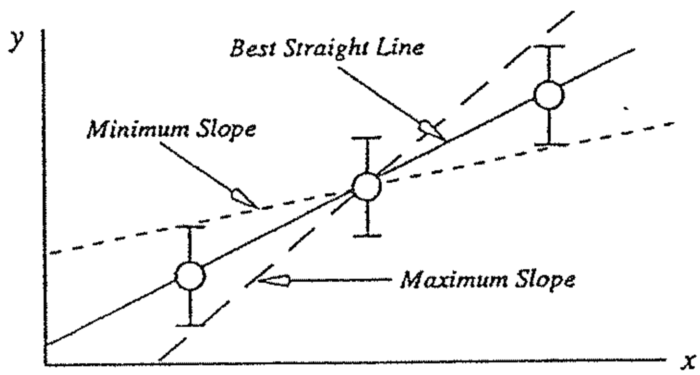
Data Point with X and Y Uncertainty Bars



Because there is not such thing as an infinitely precise data point, you should never mark a data point on a graph with just a dot. You should use a small circle, or, if relevant in size, draw an uncertainty bar.

We can easily sketch the best fit line and the two worst (the minimum slope and the maximum slope) acceptable lines by using the extremes of the uncertainty bars. The graph thus performs the function of averaging the data.

Possible Slopes Within Uncertainty Range



Uncertainty Bars for y-Values Only

Richard Feynman once said that we should be cautious with the first and last data points on any graph. He assures us that there is a reason for these points being the *first* and *last*. Therefore, they may be the least reliable points. Uncertainty bars alone may not reveal this weakness of data.

SUM AND DIFFERENCE. Let us say you want to make some calculations of a rectangular metal plate. The length L is measured to be $36\,mm$ with an estimated uncertainty of $\pm 3\,mm$, and the width W is measured to be $18\,mm$ with an estimated uncertainty of $\pm 1\,mm$. How precise will your calculations of the circumference and area be when you take into account the uncertainties?

We assume the parallel lengths and the parallel widths are identical, where $L = (36 \pm 3)\,mm$ and $W = (18 \pm 1)\,mm$. The circumference is simply the sum of the four sides.

$$C_{absolute} = L + W + L + W$$

$$C_{absolute} = 36\,mm + 18\,mm + 36\,mm + 18\,mm = 108\,mm$$

To find the *least probable* circumference, you subtract the uncertainty from each measurement and then add the four sides. The minimum lengths and widths would be $L_{min} = (36 - 3)\,mm = 33\,mm$ and $W_{min} = (18 - 1)\,mm = 17\,mm$. The *minimum* circumference is C_{min}.

$$C_{min} = L_{min} + W_{min} + L_{min} + W_{min} = 100\,mm$$

To find the *maximum probable* circumference, you add the uncertainty to each length and width, and then you add the four sides together, where $L_{max} = (36 + 3)\,mm = 39\,mm$ and $W_{max} = (18 + 1)\,mm = 19\,mm$.

$$C_{max} = L_{max} + W_{max} + L_{max} + W_{max} = 116\,mm$$

The *range* from maximum to minimum is the difference of these two values.

$$C_{range} = C_{max} - C_{min} = 116\,mm - 100\,mm = 16\,mm$$

This range includes both the added and the subtracted uncertainty values. The absolute value lies midway, so we divide the range in half to find the uncertainty in the circumference, ΔC.

$$\Delta C = \frac{C_{range}}{2} = \frac{16\,mm}{2} = 8\,mm$$

This is correctly expressed as $\pm\Delta C = \pm 8\,mm$, and the circumference is now written as

$$C \pm \Delta C = (108 \pm 8)\,mm.$$

We can generalize this process. **When we add quantities, we add their uncertainties.** The chances are that some of the uncertainties are 'plus' and others 'minus' and so a statistical approach to this would reduce the uncertainty range, but for most IB Physics work this *worst probable range* is satisfactory.

What about subtracting quantities? Subtraction is the same as addition except we add a negative quantity, $A + B = A - (-B)$. And the uncertainties? It would make no sense to subtract uncertainties because this would reduce the resultant value; we might even end up with zero uncertainty. Hence we add the uncertainties. There is a general rule for combining uncertainties with sums and differences. **Whenever we add or subtract quantities, we add their absolute uncertainties.** This is symbolized as follows.

Sum	$(A \pm \Delta A) + (B \pm \Delta B) = (A + B) \pm (\Delta A + \Delta B)$
Difference	$(A \pm \Delta A) - (B \pm \Delta B) = (A - B) \pm (\Delta A + \Delta B)$

PRODUCT AND QUOTIENT. Next we calculate the area of the metal plate. The *absolute* area is the product of length and width.

$$A_{absolute} = L_{absolute} \times W_{absolute} = 36\,mm \times 18\,mm = 648\,mm^2$$

The *minimum probable* area is the product of the minimum length and minimum width.

$$A_{min} = L_{min} \times W_{min} = 33\,mm \times 17\,mm = 561\,mm^2$$

The *maximum probable* area is the product of the maximum length and maximum width.

$$A_{max} = L_{max} \times W_{max} = 39\,mm \times 19\,mm = 741\,mm^2$$

The *range* of values is the difference between the maximum and minimum areas.

$$A_{range} = A_{max} - A_{min} = 741\,mm^2 - 561\,mm^2 = 180\,mm^2$$

This range is the result of both adding and subtracting uncertainties to the lengths and widths, and the absolute value is midway between these extreme values. Therefore, the uncertainty ΔA in the area is just half the range; it is added to and subtracted from the absolute area in order to find the extreme probable limits of the area.

$$\Delta A = \frac{A_{range}}{2} = \frac{180\,mm^2}{2} = 90\,mm^2$$

We can now express the area and its uncertainty to two significant figures.

$$A \pm \Delta A = 648\,mm^2 \pm 90\,mm^2 = (6.5 \pm 0.9) \times 10^2\,mm^2$$

To further develop the handling of uncertainties, let us now calculate the **percentages of uncertainties** in both the absolute length and absolute width measurements.

$$\Delta L\% = \frac{\Delta L}{L} 100\% = \frac{3\,mm}{36\,mm} 100\% = 8.33\%$$

$$\Delta W\% = \frac{\Delta W}{W} 100\% = \frac{1\,mm}{18\,mm} 100\% = 5.55\%$$

Adding these two percentages give us $8.33\% + 5.55\% = 13.88\%$. It is no coincidence that when we take this percentage of the absolute area that we get the same value of uncertainty we got when we calculate the extreme values.

$$13.88\% \times A_{absolute} = 0.1388 \times 648\,mm^2 = 89.9424\,mm^2$$

The calculated area is now $A \pm \Delta A = (648 \pm 89.9424)\,mm^2 \approx (6.5 \pm 0.9) \times 10^2\,mm^2$. Expressed with a percentage of uncertainty, the area is written as $A \pm \Delta A\%$.

$$A \pm \Delta A\% = 648\,mm^2 \pm 13.88\% \approx 650\,mm^2 \pm 14\%$$

The only apparent difference between finding the *extreme values* and using *percentages* is that using percentage is easier; it requires fewer calculations. Moreover, using percentages will simplify our work when dealing with complex equations involving variations of product and quotient, including square roots, cubes, etc. Although some examples may show a slight difference between the calculation of extremes and the use of percentages, most of this difference is lost when rounding off to the correct number of significant figures. However, when making higher level calculation (such as cubes and square roots), any slight difference between the two methods may become noticeable, and the use of percentages will yield a smaller uncertainty range. This, of course, is desirable.

When multiplying quantities, we add the percentages of uncertainties to find the uncertainty in the product. But what about quotients? Dividing two quantities is the same as multiplying one by the reciprocal of the other, such that $(A/B) = A(1/B)$. This means we should add the percentages of uncertainties when we divide. You might think that you could subtract percentages of uncertainties when dividing but then you would be reducing the effective uncertainty and you might end up with zero or even negative uncertainty. This is not acceptable. Therefore, the rule of products and for quotients is one and the same. **We add the percentages of uncertainties when we find the product or quotient of two or more quantities.** This rule is symbolized as follows.

Product $$(A \pm \Delta A) \times (B \pm \Delta B) = (A \times B) \pm \left[\left(\frac{\Delta A}{A}100\%\right) + \left(\frac{\Delta B}{B}100\%\right)\right]$$

Quotient $$\frac{A \pm \Delta A}{B \pm \Delta B} = \frac{A}{B} \pm \left[\left(\frac{\Delta A}{A}100\%\right) + \left(\frac{\Delta B}{B}100\%\right)\right]$$

For example, find z where $z = x/y$ and $x \pm \Delta x = 22 \pm 1$ and $y \pm \Delta y = 431 \pm 9$.

$$z \pm \Delta z = \frac{x \pm \Delta x}{y \pm \Delta y} = \frac{22 \pm 1}{431 \pm 9} = \frac{22}{431} \pm \left[\left(\frac{1}{22}100\%\right) + \left(\frac{9}{431}100\%\right)\right]$$

$$z \pm \Delta z = \frac{22}{431} \pm (4.545\% + 2.089\%) = 0.05104 \pm 6.633\%$$

$$z \pm \Delta z = 0.05104 \pm 0.00338 = (5.1 \pm 0.3) \times 10^{-2}$$

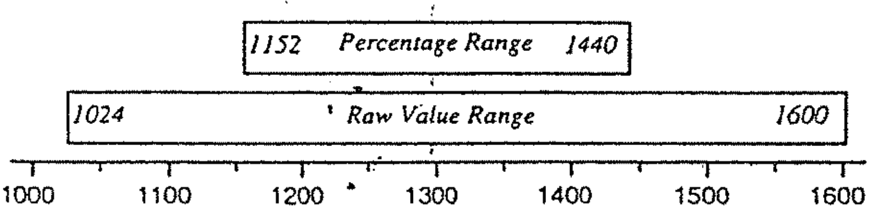
IGNORING AN UNCERTAINTY. If one of the percentage uncertainties is less than a quarter of the other, you may ignore the smaller percentage. For instance, $4.8\% + 0.9\% \approx 4.8\%$ and not 5.7%. A statistical approach would show that uncertainties are not as bad as the combining rule suggests, but statistical methods are beyond the needs of most student work. So, when one percentage is significantly smaller than the other we can ignore it. I say a quarter arbitrarily but when taking significant figures into account, this suggestion is justified.

POWERS. Multiplication can be seen as equivalent to addition. For example, 3×4 is the sum of three 4s: $3 \times 4 = 4 + 4 + 4$. **Squaring** a number, then, is the same as multiplying it by itself, $A^2 = A \times A$. With uncertainties, $(A \pm \Delta A)^2 = (A \pm \Delta A)(A \pm \Delta A)$. Where the uncertainty in A is expressed as a percentage, we can see that this percentage is added to itself to find the percentage of uncertainty in A^2.

$$(A \pm \Delta A\%)^2 = (A \pm \Delta A\%)(A \pm \Delta A\%) = A^2 \pm 2\Delta A\%$$

Often the range of uncertainty is greatly reduced by using the method of percentages compared to the method of calculating upper and lower limits. Consider the number line for $(x \pm \Delta x)^2$ where $x \pm \Delta x = 36 \pm 4$. The percentage range is ± 144, while the upper and lower limits when calculated by the maximum and minimum probable values of x are -272 and $+304$.

NUMBER LINE: $x^2 = 36^2 = 1296$



Similarly, when you **cube** a quantity you add the percentage of uncertainty of the quantify to itself three times, because $A^3 = A \times A \times A$. Hence $(A \pm \Delta A\%)^3 = A^3 \pm 3\Delta A\%$. We can generalize the *adding of percentages* rule for any power, $A^n = A_1 + A_2 + \ldots + A_n$. First, we find the percentage of uncertainty in A, which is $\Delta A\%$, and then we add this to itself n times in order to find the percentage of uncertainty in A^n. The rule for any n^{th} power is symbolized as follows.

n^{th} *Power*	$(A \pm \Delta A)^n = A^n \pm n\left(\frac{\Delta A}{A} 100\%\right) = A^n \pm n\Delta A\%$

For instance, if x is known to 3% then x^2 is known to 6%, or x^5 is known to 15%, and so on. Using percentages reduces the uncertainty range when compared to calculating raw values of maximums and minimums.

ROOTS. Taking the **square root** of a number is the same as taking that number to a power of one-half, $\sqrt{x} = x^{\frac{1}{2}}$, where, for instance, $\sqrt{36} = 36^{\frac{1}{2}} = 6$. We have to be cautious here. Previously, mathematical operations increased the overall uncertainty. But taking the square root of a number yields a much smaller number, and we would not expect the uncertainty to increase to perhaps to a value greater than the square root itself. Also, our combination of uncertainties must be consistent in a way that when you reverse the process you end up with the same number and with the same uncertainty you started with. For instance, $\sqrt[3]{x^3}$ must equal x, and the same with the uncertainties. Therefore it seems fair to say that taking x to the one-half power will cut the uncertainty in half. Then, when you square the square root and propagate the uncertainties, you end up with the original uncertainty and the original absolute value.

With an argument similar to square roots, we can say that the **cube root** will reduce the percentage of uncertainty to one-third, or the fourth root reduces the percentage of uncertainty to one-fourth. In general, where $\sqrt[n]{A} = A^{\frac{1}{n}}$, we can state a rule. **The n^th^ root reduces the percentage of uncertainty by *1/n*.** This is symbolized as follows.

n^{th} *Root*	For $\sqrt[n]{A \pm \Delta A}$, we find $\sqrt[n]{A} \pm \frac{1}{n}\left(\frac{\Delta A}{A} 100\%\right) = \sqrt[n]{A} \pm \frac{\Delta A\%}{n}$

PURE NUMBERS. Often calculations involve numbers such as 2, ½, ⅔, or π. These are pure numbers and have no uncertainty associated with them. If you were to measure the thickness of 100 sheets of paper to be $(26 \pm 1)mm$ then you could say that the thickness of a single sheet is $(0.26 \pm 0.01)mm$. This is reasonable because if you reversed the process and you measured one sheet as $(0.26 \pm 0.01)mm$ and then multiplied by 100, you would end up with the same value and uncertainty as measuring 100 sheets. The general rule for propagating uncertainties when using pure numbers is straight forward. **When you multiply or divide by pure numbers you multiply or divide the absolute uncertainty by the pure number.**

For example, when you measure the diameter of a circular object and want to calculate the radius, where $r = d/2$ and $d \pm \Delta d = (4.6 \pm 0.2)cm$, you find that dividing by 2 will cut the uncertainty in half. Therefore $r \pm \Delta r = (2.3 \pm 0.1)cm$. A circle with this radius will then have a circumference $c = 2\pi r$ and uncertainty $\Delta c = 2\pi \Delta r$ as follows.

$$c \pm \Delta c = 2\pi (r \pm \Delta r) = 2\pi (2.3 \pm 0.1)cm$$

$$c \pm \Delta c = 14.4513cm \pm 0.6283cm = (14 \pm 1)cm$$

FUNCTIONS. You must not assume that the adding percentage rule is applicable to all calculations. There are certain mathematical functions (such as reciprocals, logarithms and trigonometric functions) where, if the uncertainty is more than a few percent, we should use the absolute uncertainty to compute upper and lower limits which then determine the uncertainty in the function value. Why? The percentages are necessarily symmetrical, $\pm\Delta x\%$; the same uncertainty value Δx is added and subtracted from the absolute value to find the probable range. A more accurate description of the range of uncertainty with, say, the reciprocal function, is that it is not symmetrical.

Take, for example, the **reciprocal** of x where $y = (1/x)$ and $x \pm \Delta x = 0.36 \pm 0.06$. Calculating percentages, the uncertainty range is ± 0.46 on either side of y where $y = 2.78$. Calculating the minimum and maximum values of y gives 0.40 below and 0.56 above the absolute value of y, which is an asymmetry with a range shifted upwards. We can state the following rule. **The uncertainty range for the reciprocal function is found by calculating the upper and lower limits using minimum and maximum raw values of the denominator.** We should not use percentages. This is especially true when drawing uncertainty bars on graphs because it is the range of probable values that allows you to draw the best straight line and we don't want this shifted symmetrically when in fact the correct range is

asymmetrical. However, when uncertainties are small (a few percent or less), we may assume that the percentage of the uncertainty in x is the same as in y, where y is the reciprocal of x.

A similar asymmetry in raw value uncertainty distribution occurs when taking **logarithms**. This asymmetry is especially important when we graph functions because when we attempt to draw a best straight line we need to pay attention to the larger side of the uncertainty for a given data point. Logarithms, however, have the asymmetry in the opposite direction (shifted to a lower range) compared to the reciprocal function. **When calculating the uncertainty range of a quantity that is expressed in logarithms, always find the uncertainty range using minimum and maximum raw values.**

The **trigonometric functions** of sine, cosine and tangent are examples of pure functions. The uncertainty range here can be calculated using minimum and maximum values. Try an example of a large uncertainties and show yourself that using percentages yields a greater uncertainty range than calculating minimum and maximum absolute values. To find the refractive index n using a measured critical angle θ with the equation $n = 1/\sin\theta$, you should make three calculations, $\sin(\theta - \Delta\theta)$, $\sin(\theta + \Delta\theta)$, and $\sin\theta$. **When using trigonometric functions it is best to determine the uncertainty by calculating upper and lower limits using absolute values.**

EXAMPLE CALCULATION. The following illustrates what a you might do when calculating gravity from a simple pendulum experiment. The pendulum length and period are measured as $\ell = 2.54 \pm 0.02\,m$ and $T = 3.2 \pm 0.1\,s$. Theory tells us that $T = 2\pi\sqrt{\ell/g}$. First, we solve for gravity and then we find the percentages of uncertainties.

$$g = \frac{4\pi^2 \ell}{T^2} = \frac{4\pi^2(2.54\,m)}{(3.2\,s)^2} = 9.792498\,m\,s^{-2}$$

$$\Delta\ell\% = \frac{0.02\,m}{2.54\,m}100\% = 0.7874\%$$

$$\Delta T\% = \frac{0.1\,s}{3.2\,s}100\% = 3.125\%$$

$$\textit{Period Squared Uncertainty} = 2\times\Delta T\% = 2\times 3.125\% = 6.2500\%$$

When we divide the length by the period squared, we add the percentages of uncertainties, $0.7874\% + 6.2500\%$. But here we recall the general rule that if one of the percentages is a quarter or less than the other, we can ignore the smaller percentage. Therefore, we can say that the uncertainty in this problem is 6.2500%. We now change the percentage of uncertainty into an absolute value, and then express the calculated value of gravity to two significant figures.

$$g = 9.792498\, m s^{-2} \pm 0.612031\, m s^{-2} \approx (9.8 \pm 0.6) m s^{-2}$$

ADVANCED STUDENT WORK. The rules and suggestions in this booklet relate to the IB Physics curriculum. When doing more specialized work, however, such as an extended essay in physics or a more developed class lab, you may want to use more sophisticated methods, including statistical analysis. When combining two uncertainties, $\pm\Delta A$ and $\pm\Delta B$, it was suggested that we simply add the absolute uncertainties. There are four combinations here: $+\Delta A$ with $+\Delta B$, $+\Delta A$ with $-\Delta B$, $-\Delta A$ with $+\Delta B$, and $-\Delta A$ with $-\Delta B$. This means that there is a 50% chance that combining uncertainties will actually reduce the overall uncertainty. The rule of adding is really for the worst possible case. Here, then, are two advanced rules that refine the combination of uncertainties.

For **sum** and **difference**, the resultant uncertainty is the square root of the sum of the squares of the individual uncertainties. This improves upon the uncertainty range (i.e., it reduces the range) when compared to simply adding the absolute uncertainties.

$$\Delta x = \sqrt{\Delta A^2 + \Delta B^2 + \Delta C^2 + etc.}$$

For **product** and **quotient**, the fractional uncertainty is the square root of the sum of the squares of the individual fractional uncertainties. This ratio can be turned into a percentage by multiplying it by 100%. This improves the uncertainty range by reducing it.

$$\frac{\Delta x}{x} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta C}{C}\right)^2 + etc.}$$

CONCLUSION. Because scientific theories are simplified approximations of a complex reality, an appreciation of the precision and accuracy of experimental results is an essential part of the experimental process. The I.B. Physics Course Guide tells us that *students are to gain an understanding of what it means to determine a value, and the confidence that one can associate with that value. Since physics is a quantitative science at its most fundamental level, this is an essential skill.*

SUMMARY OF BASIC RULES

Sum $$(A \pm \Delta A) + (B \pm \Delta B) = (A + B) \pm (\Delta A + \Delta B)$$

Difference $$(A \pm \Delta A) - (B \pm \Delta B) = (A - B) \pm (\Delta A + \Delta B)$$

Product $$(A \pm \Delta A) \times (B \pm \Delta B) = (A \times B) \pm \left[\left(\frac{\Delta A}{A} 100\%\right) + \left(\frac{\Delta B}{B} 100\%\right)\right]$$

Quotient $$\frac{A \pm \Delta A}{B \pm \Delta B} = \frac{A}{B} \pm \left[\left(\frac{\Delta A}{A} 100\%\right) + \left(\frac{\Delta B}{B} 100\%\right)\right]$$

n^{th} Power $$(A \pm \Delta A)^n = A^n \pm n\left(\frac{\Delta A}{A} 100\%\right) = A^n \pm n \Delta A\%$$

n^{th} Root For $\sqrt[n]{A \pm \Delta A}$, we find $\sqrt[n]{A} \pm \frac{1}{n}\left(\frac{\Delta A}{A} 100\%\right) = \sqrt[n]{A} \pm \frac{\Delta A\%}{n}$

PROBLEMS

1. The diameter of a wire is measured repeatedly in different places along its length. The measurements in millimeters are 1.26, 1.26, 1.29, 1.31, 1.28, 1.27, 1.26, 1.25, 1.28, 1.32, 1.21, 1.27, 1.22, 1.29 and 1.28. What is the average of these measurements? Express your answer with an uncertainty.

2. Given two masses, $m_1 = (100.0 \pm 0.4)g$ and $m_2 = (49.3 \pm 0.3)g$, what is their sum, $m_1 + m_2$, and what is their difference, $m_1 - m_2$, both expressed with uncertainties?

3. You drive your car at a constant speed such that the reading on the speedometer is 40 mph. The speedometer is assumed to be accurate to ±2 mph. At the end of your drive you would like to know how far you have driven but you forgot to look at the mileage. You assume that you have been driving for about 4 hours, give or take 15 minutes. Estimate how far you traveled and assign an uncertainty to your answer.

4. What is the uncertainty in the calculated area of a circle whose radius is determined to be $r = (14.6 \pm 0.5)cm$?

5. What is the uncertainty in the calculated density $(density = mass/volume)$ of a block of mass $m = (245 \pm 2)g$ and with sides of $(2.5 \pm 0.1)cm$, $(4.8 \pm 0.1)cm$ and $(10.2 \pm 0.1)cm$?

6. An electrical resistor has a 2% tolerance and is marked $R = 1800\Omega$. What is the range of acceptable values that the resistor might have? An electrical current of $I = (2.1 \pm 0.1)mA$ flows through the resistor. What is the uncertainty in the calculated voltage across the resistor where the voltage is given as $V = IR$?

7. An accelerating object has an initial speed of $u = (12.4 \pm 0.1)ms^{-1}$ and a final speed of $v = (28.8 \pm 0.2)ms^{-1}$. The time interval for this change in speed is $\Delta t = (4.2 \pm 0.1)s$. Acceleration is defined as $a = (v - u)/\Delta t$. Calculate the acceleration and its uncertainty.

8. The distance s that an accelerating body covers in a certain amount of time t is given by the equation $s = mt^2$ where the constant $m = 16ms^{-2}$. Compare the distances traveled (and the uncertainty) for two time intervals: $t_1 = (3.0 \pm 0.1)s$ and $t_2 = (9.0 \pm 0.1)s$.

9. What is the volume and its uncertainty for a sphere with a radius of $r = (21 \pm 1)\,mm$?

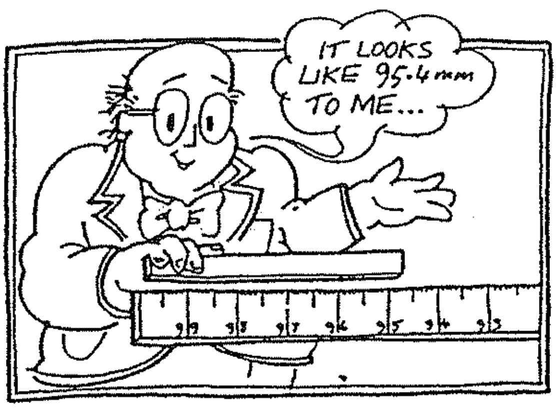
10. Frequency and period are related as reciprocals., What is the period and its absolute uncertainty when the frequency of $1\,kHz$ is known to 2%?

11. Angle θ is known to about $\pm 1°$. If $\angle\theta = 34°$ what are the minimum and maximum acceptable values for $\tan\theta$?

12. Einstein's famous equation relates energy and mass with the square of the speed of light, where $E = mc^2$. What is the percentage of uncertainty and the absolute uncertainty of the energy for a mass $m = 1.00\,kg$ where the speed of light is $c = 3.00 \times 10^8\,ms^{-1}$?

13. With a good stopwatch and some practice, one can measure times ranging from about a second up to many minutes with an uncertainty of 0.1 second or so. Suppose that we wish to find the period τ of a pendulum with $\tau \approx 0.5\,s$. If we time one oscillation, we will have an uncertainty of about 20%, but by timing several successive oscillations, we can do much better.

    If we measure the time for five successive oscillations and get $2.4 \pm 0.1\,s$, what is the final answer (with an absolute uncertainty) for the period? What if we measure 20 oscillations and get a time of $9.4 \pm 0.1\,s$?



The End.